

Classifying Asynchronous Elementary Cellular Automata with the 0–1 Test for Chaos

Pietro Di Lena¹[0000–0002–1838–8918] and Nicola Travaglini¹[0009–0006–1030–4915]

Department of Computer Science, University of Bologna, Bologna, Italy
`pietro.dilena@unibo.it`
`nicola.travaglini3@studio.unibo.it`

Abstract. We investigate the effect of asynchronous updating on the dynamical behavior of elementary cellular automata (ECA) by means of the 0–1 test for chaos. While the classical theory of cellular automata is based on synchronous evolution, many natural systems are inherently asynchronous. It is therefore interesting to understand how asynchrony affects their qualitative dynamics, in particular whether it induces transitions from chaotic to regular behavior or vice versa. We adopt the experimental framework introduced in [6] and extend it to fully asynchronous ECA (AECA): for each rule, we compute the 0–1 test statistic and compare the resulting classification with both its synchronous counterpart and Wolfram’s empirical classification [8]. Our results show that asynchronous updating can significantly alter the dynamical behavior of cellular automata. In particular, several rules that are regular in the synchronous setting exhibit chaotic behavior under asynchronous updates, while some chaotic rules become regular.

Keywords: Elementary Cellular Automata · Asynchronous Cellular Automata · Chaotic Dynamics · 0–1 test for chaos.

1 Introduction

Cellular automata (CA) [4] provide a simple yet powerful framework for studying complex dynamical phenomena, including different notions of chaos. Despite their simple local rules, CA can exhibit behaviors ranging from regular and predictable evolution to highly irregular and chaotic dynamics [8].

In this work, we investigate how the dynamical behavior of cellular automata changes when passing from synchronous to asynchronous updating schemes [2]. In particular, we study *state shifts*, namely situations in which a rule that is regular under synchronous evolution becomes chaotic under asynchronous dynamics, or conversely. While several formal definitions of chaos exist for synchronous CA [1, 5], no widely accepted notion of chaos is currently available for asynchronous cellular automata (ACA).

To address this problem, we extend the experimental framework introduced by Terry-Jack and O’Keefe [6], who applied the 0–1 test for chaos to elementary cellular automata (ECA) and compared the resulting classification with Wolfram’s empirical classification of ECA dynamics [8]. The 0–1 test is a simple and

computationally efficient method for distinguishing between regular and chaotic dynamics from time series data [3]. We apply this methodology to fully asynchronous ECA (AECA) and compare the resulting classifications with those obtained in the synchronous setting. Our results show that asynchronous updating can significantly modify the dynamics of cellular automata: several rules that are regular in the synchronous case become chaotic under asynchronous evolution, while some chaotic rules become regular or exhibit intermediate behavior.

The paper is organized as follows. Section 2 reviews the necessary background on CA, ECA, and AECA. Section 3 introduces the 0–1 test for chaos. Section 4 describes the experimental setup, while Section 5 presents the results. Finally, Section 6 concludes the paper and discusses future work.

2 Cellular Automata

Cellular automata (CA) are discrete dynamical systems defined on regular lattices, consisting of a collection of cells that evolve in discrete time according to local interaction rules. Formally, let A be a finite set, called the alphabet, and consider the configuration space $A^{\mathbb{Z}}$ of bi-infinite sequences $(x_i)_{i \in \mathbb{Z}}$ where $x_i \in A$. A (one-dimensional) cellular automaton is a map

$$F : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$$

defined by a *local rule* $f : A^{2r+1} \rightarrow A$ of finite *radius* $r \in \mathbb{N}$ such that

$$F(x)_i = f(x_{i-r}, \dots, x_{i+r}) \quad \text{for all } i \in \mathbb{Z}.$$

The *shift map* $\sigma : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is a cellular automaton defined by $\sigma(x)_i = x_{i+1}$. A fundamental result, known as the Curtis–Hedlund–Lyndon theorem [4], states that a map $F : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is a cellular automaton if and only if it is *shift-invariant*, i.e., $F \circ \sigma = \sigma \circ F$, and *continuous* with respect to the product topology on $A^{\mathbb{Z}}$.

In the following subsections we introduce the classes of elementary cellular automata and asynchronous cellular automata.

2.1 Elementary Cellular Automata

A particularly important class of one-dimensional cellular automata is given by the elementary cellular automata (ECA), introduced by Wolfram [7]. In this case, the alphabet is binary, $A = \{0, 1\}$, and the radius is $r = 1$, so that the local rule

$$f : \{0, 1\}^3 \rightarrow \{0, 1\}$$

depends on the state of a cell and its two nearest neighbors. Since there are $2^3 = 8$ possible neighborhood configurations, there are $2^8 = 256$ distinct ECA rules, each identified by an integer obtained by encoding the outputs of the local rule.

Many ECA rules exhibit equivalent dynamics under simple transformations, namely mirroring (left-right reflection) and complementation ($0 \leftrightarrow 1$). Taking

these symmetries into account, the 256 rules can be grouped into 88 equivalence classes, and it is common to consider only one representative from each class (typically the one associated to the smallest integer).

Despite their simple definition, ECA exhibit a wide range of dynamical behaviors, from simple fixed points to highly complex and chaotic patterns [8]. Wolfram proposed an empirical classification into four qualitative classes based on long-term behavior from random initial conditions:

- **Class I:** evolution leads to homogeneous configurations,
- **Class II:** evolution leads to periodic or stable structures,
- **Class III:** evolution exhibits chaotic behavior,
- **Class IV:** evolution produces complex structures with localized interactions.

The classification of the 88 representative rules is reported in Table 1. Although widely used, this classification is largely empirical and does not admit a fully rigorous mathematical characterization.

2.2 Asynchronous Cellular Automata

In the standard formulation, cellular automata evolve synchronously, meaning that all cells are updated simultaneously at each time step. Asynchronous cellular automata (ACA) relax this assumption by allowing cells to be updated at different times [2].

Formally, at each time step a subset of sites $U \subseteq \mathbb{Z}$ is selected, and the local rule is applied only to the cells in U , while the others remain unchanged. Denoting by F_U the corresponding partial update, the evolution is given by

$$x^{(t+1)} = F_{U_t}(x^{(t)}),$$

where $\{U_t\}_{t \geq 0}$ is a sequence of update sets.

Several update schemes have been studied, including fully asynchronous (single-site random updates), α -asynchronous (each site updated independently with probability α), and block-sequential or deterministic schedules.

Unlike synchronous CA, ACA do not define a single deterministic global map but rather a family of possible evolutions depending on the update sequence. As a consequence, their dynamics are typically described in probabilistic terms and may differ significantly from their synchronous counterparts [2]. In this work, we focus exclusively on fully asynchronous ECA (AECA).

3 The 0–1 Test for Chaos

The 0–1 test for chaos, introduced by Gottwald and Melbourne [3], is a binary test for distinguishing between regular and chaotic dynamics directly from a scalar time series. Unlike Lyapunov-based approaches, it does not require phase space reconstruction and is therefore particularly suitable for systems where only observational data are available.

Given a real-valued time series $\phi(n)$, the test defines the transformations

$$p_c(n) = \sum_{j=1}^n \phi(j) \cos(jc), \quad q_c(n) = \sum_{j=1}^n \phi(j) \sin(jc),$$

where $c \in (0, 2\pi)$ is an angle parameter. The sequence $\{(p_c(n), q_c(n))\}_{n \geq 1}$ can be interpreted as a trajectory in the plane. The growth of this trajectory is measured through the mean square displacement

$$M_c(n) = \frac{1}{N-n} \sum_{j=1}^{N-n} [(p_c(j+n) - p_c(j))^2 + (q_c(j+n) - q_c(j))^2], \quad (1)$$

where N is the trajectory length and $n \ll N$. For regular dynamics, the trajectory $(p_c(n), q_c(n))$ remains bounded and $M_c(n)$ grows sublinearly, whereas for chaotic systems it exhibits diffusive behavior and grows approximately linearly with n . The corresponding growth rate is estimated through the correlation coefficient

$$K_c = \text{corr}(\log n, \log M_c(n)), \quad (2)$$

with the convention

$$\log M_c(n) = \begin{cases} \log M_c(n) & \text{if } M_c(n) > 0, \\ 0 & \text{if } M_c(n) = 0. \end{cases}$$

To improve robustness, the computation is repeated for multiple values of the angle parameter c , and the final statistic is defined as

$$K = \text{median}_{c \in (0, 2\pi)} K_c.$$

Values $K \approx 0$ indicate regular dynamics, while $K \approx 1$ indicate chaotic behavior. Intermediate values (typically $K \approx 0.5$) are often associated with weak chaos or “edge of chaos” dynamics.

4 Experimental setup

We adopt the experimental framework of [6], where the 0–1 test for chaos is applied to the 88 representative elementary cellular automata.

In the synchronous setting, the configuration width is fixed to $W = 637$ (as in [8]), the trajectory length is set to $N = 50,000$, and the maximum displacement parameter is chosen as $n = N/3$. A fixed initial configuration $x \in \{0, 1\}^W$ with approximately balanced density is used. Each configuration is converted into a scalar observation through the Gray encoding

$$\phi(x) = \frac{1}{2^W} \sum_{w=1}^W 2^{W-w} (x_w \oplus x_{w+1}),$$

assuming periodic boundary conditions. The statistic K is computed using five values of the angle parameter c , given by $c_i = \frac{i\pi}{20}, i = 1, \dots, 5$.

For fully asynchronous ECA (AECA), we retain the same parameters with a few modifications. Time is measured in Monte Carlo Steps (MCS), where one MCS corresponds to W independent single-site updates. Trajectories have length $N = 50,000$ MCS. For each rule, 20 independent trajectories are generated using different pseudo-random seeds, and the final value of K is computed as the average over these realizations.

Overall, this corresponds to a total of $88 \times 20 = 1,760$ trajectories, each consisting of 31,850,000 micro-steps.

5 Effect of Asynchronous Updating on ECA Dynamics

We compare the behavior of ECA and AECA with respect to the Wolfram classification in order to identify *state shifts*, i.e., transitions from regular to chaotic behavior (or vice versa) induced by asynchronous updating.

The results are summarized in Table 1. The ECA classification is taken from [6], while the AECA classification is obtained in this work. The labels *Regular*, *Chaotic*, and *Edge of chaos* correspond respectively to $K \approx 0$, $K \approx 1$, and intermediate values (i.e., $K \approx 0.5$).

W. Class	Rules	0–1 Test ECA	0–1 Test AECA
Class I	0, 8, 32, 40, 128, 136, 160, 168	Regular	Regular
Class II	2, 4, 5, 10, 12, 13, 24, 26, 28, 29, 34,	Regular	Regular
	36, 42, 44, 50, 56, 58, 72, 74, 76, 77,		
	78, 94, 104, 108, 130, 132, 140, 162,		
	164, 172, 200, 204, 232		
	23, 152, 178, 184		
	1, 3, 6, 7, 9, 11, 14, 15, 19, 25, 27,	Regular	Edge of chaos Chaotic
	33, 35, 37, 38, 43, 46, 51, 57, 62, 73,		
	134, 138, 142, 154, 156, 170		
Class III	18	Regular	Regular
	126	Regular	Chaotic
	22	Chaotic	Edge of chaos
	30, 45, 60, 90, 105, 122, 146, 150	Chaotic	Chaotic
Class IV	41	Regular	Chaotic
	106	Chaotic	Regular
	54, 110	Chaotic	Chaotic

Table 1: Comparison between synchronous (ECA) and asynchronous (AECA) dynamics under the Wolfram classification.

Several interesting phenomena emerge from Table 1. Many rules that are regular in the synchronous setting become chaotic under asynchronous updates, particularly within Class II. Conversely, few chaotic rules (e.g., Rules 22 and 106)

exhibit reduced chaotic behavior under asynchronous dynamics. These observations indicate that asynchrony can both enhance and suppress chaos, depending on the rule.

In the following subsections, we discuss representative examples of the two main classes of state shifts: regular-to-chaotic transitions and chaotic-to-regular transitions.

5.1 *Regular* $\rightarrow$ *Chaotic*

This is the most common type of state shift: 29 rules transition from regular to chaotic behavior under asynchronous dynamics. Interestingly, 27 of these rules belong to Wolfram Class II, which consists of cellular automata exhibiting periodic or stable behavior. Consistently, all synchronous ECA in Class II are classified as regular by the 0–1 test.

As representative examples, we compare Rule 5 (no state shift) and Rule 1 (regular $\rightarrow$ chaotic), which differ only in the transition associated with the neighborhood 010:

$$\begin{aligned} \text{Rule 5: } 000 &\mapsto 1, \quad 010 \mapsto 1, \quad \text{otherwise } 0, \\ \text{Rule 1: } 000 &\mapsto 1, \quad 010 \mapsto 0, \quad \text{otherwise } 0. \end{aligned}$$

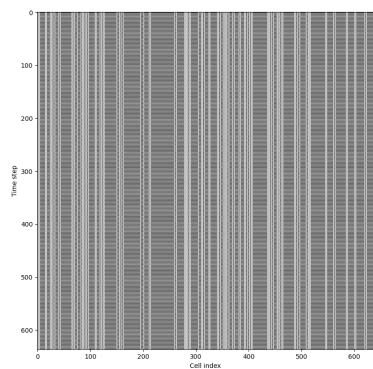
Under synchronous dynamics, both rules exhibit simple periodic behavior. Under asynchronous updates, however, their dynamics diverge significantly. Rule 5 rapidly stabilizes (Fig. 1(c)–(d)), whereas Rule 1 becomes highly dependent on the update order, producing irregular trajectories (Fig. 1(a)–(b)). This example illustrates how a minimal modification of the local rule can dramatically affect the robustness of the dynamics under asynchronous updating.

5.2 *Chaotic* $\rightarrow$ *Regular*

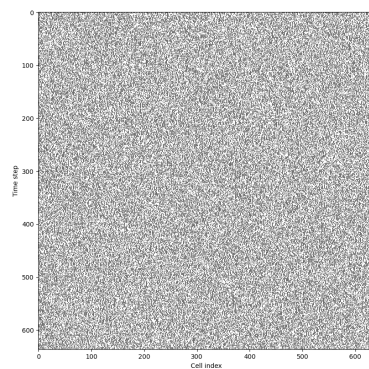
Rule 106 in Wolfram Class IV is the only rule that shifts from chaotic to regular behavior under asynchronous dynamics. While synchronous evolution produces complex patterns with long-lived interactions (Fig. 2(a)), asynchronous updates rapidly destroy these structures, causing the system to converge to a stable configuration (Fig. 2(b)). This suggests that the complexity of Rule 106 strongly depends on synchronization, and that asynchronous dynamics can suppress complex dynamics by disrupting the temporal coherence of the system.

6 Conclusions

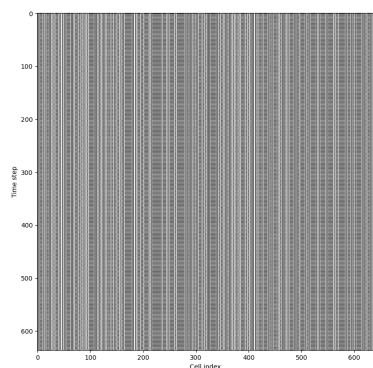
In this work, we investigated the effect of asynchronous updating on the dynamical behavior of elementary cellular automata (ECA) using the 0–1 test for chaos. By extending the experimental framework of [6] to fully asynchronous cellular automata (AECA), we performed a systematic comparison between synchronous and asynchronous dynamics across the 88 representative ECA rules.



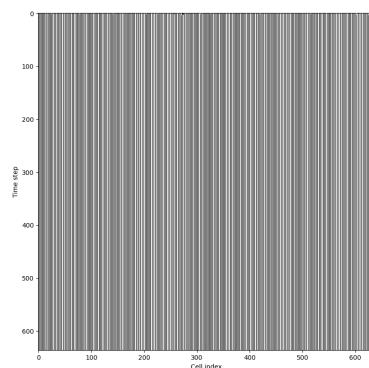
(a) Rule 1 synchronous trajectory.



(b) Rule 1 asynchronous trajectory.



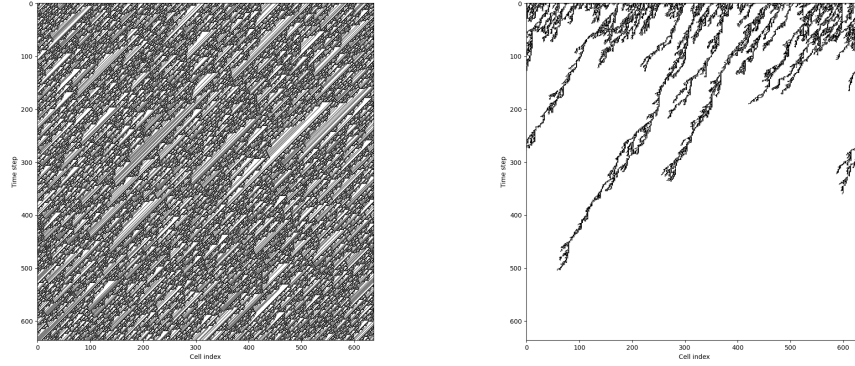
(c) Rule 5 synchronous trajectory.



(d) Rule 5 asynchronous trajectory.

Fig. 1: Comparison of Rules 1 and 5 under synchronous and asynchronous dynamics.

Our results show that asynchronous updating can significantly alter the qualitative behavior of cellular automata. In particular, we identified several instances of *state shifts*, where rules transition from regular to chaotic behavior or vice versa. These findings highlight the strong dependence of dynamical properties on the update scheme and suggest that chaoticity in cellular automata is not solely determined by the rule, but also by the temporal structure of the evolution. Moreover, while many rules exhibit substantial changes under asynchronous dynamics, some classes remain robust, consistently displaying the same qualitative behavior.



(a) Rule 106 synchronous trajectory. (b) Rule 106 asynchronous trajectory.

Fig. 2: Comparison of Rule 106 trajectories.

Several directions for future work can be considered. First, it would be interesting to extend this analysis to different asynchronous updating schemes beyond the fully asynchronous model considered here. Second, a systematic comparison with more formal classifications, such as the equicontinuity classification and Devaney's notion of chaos, would provide deeper insight into the relationship between numerical and topological notions of complexity. Finally, further investigation of the sensitivity of the 0–1 test to its parameters (e.g., the choice of the angle parameter c , trajectory length, and transient removal) could help improve its reliability when applied to cellular automata.

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